QUESTIONS AND DISCUSSIONS.

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DISCUSSIONS.

In the first discussion this month Professor Moritz shows a method of deriving the area bounded by a parabola and certain straight lines by principles of elementary character such as are used in obtaining the area of a circle in plane geometry. The method applies with equal ease to the ordinary quadratic parabola and to parabolas of higher order. It is not likely that this proof will be entirely intelligible to all the students of an ordinary high-school class; but the same statement holds in connection with the area of the circle. It is worthy of note that the validity of Professor Moritz's proof depends on the possibility of saying that the quantity which he calls F^2 remains finite for all choices of r and n; this is easy to see on writing out the value of F^2.

Mr. Haldeman shows how to solve a cubic equation graphically by means of ruler, compass, and an appropriately chosen equilateral hyperbola, and applies the result to the graphical construction of the side of a regular heptagon inscriptible in a given circle. Since equilateral hyperbolas differ only by translations, rotations, and similarity transformations, it seems possible that the construction could be performed by the use of ruler, compass, and a given fixed equilateral hyperbola.

Professor Schmiedel indicates a method for obtaining the sum of a definite number of terms of certain types of series, and gives some interesting interpretations of the results. All the series considered are of frequent occurrence in connection with Fourier series and other allied developments.

The fourth discussion is a short note by Mr. M. W. Jacobs on the reason for the occasional success of a false rule for finding the hypotenuse of a right triangle in terms of the two sides. The case actually considered, even in the extended form of the corollary, is so simple as to be almost obvious. If we ask when it is possible for the hypotenuse to be represented linearly with rational coefficients (neither being required to be unity) in terms of the perpendicular sides, we have a slightly more complicated case, which leads to a familiar type of Diophantine equation. May we have the discussion of this case also?

I. On the Quadrature of the Parabola.

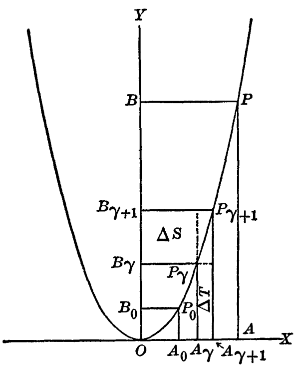
By R. E. Moritz, University of Washington.

While most books on analytical geometry derive the formula for the area of an ellipse from its relation to the area of the circumscribed circle, the attempt is but seldom made to derive the formula for the area of a segment of the parabola.

The reason for this must be sought in the fact that the classic proofs of the formula in question require principles which are foreign to the methods of analytical geometry. Thus the famous proof by Archimedes involves, besides ten or eleven preliminary propositions, the sum of the infinite series $\Sigma(1/n^2)$. On the other hand, the method of infinitesimals (indivisibles) first applied to the quadrature of the parabola by Cavalieri and later extended by Wallis to the quadrature of the curve $y = x^p$, involves the limit of the sum $(1^p + 2^p + 3^p + \cdots + n^p)/(n^{p+1})$ as n approaches infinity.

The following proof is probably not new though the writer is not aware that it is recorded in any of the sources available to him. It is certainly far more simple and direct than any of the classic proofs. In fact it involves no principles except such as are familiar to every student of elementary algebra and geometry. Even when extended to the quadrature of the general parabola, $y = px^m$, it comes well within the range of the average high-school student.

Quadrature of the Parabola $y = px^m$. Let $P_0(x_0, y_0)$, $P(x, y)$, represent any two points on the parabola which lie on the same side of the vertex O, and let it be required to determine the area included between the curve, either axis, and the perpendiculars from these points on the y-axis.



From P_0 and P draw the perpendiculars P_0A_0 and PA to the x-axis. Divide A_0A into n equal parts and call the length of each part h. At the points of division, A_r, erect perpendiculars and through the points P_r, in which these perpendiculars meet the curve, draw lines parallel to the x-axis intersecting the y-axis in the points B_r. Denote the area B_0P_0PB by S, the area APP_0A_0 by T; the area of the rectangle P_rB_{r+1} by ΔS, and the area of the rectangle P_rA_{r+1} by ΔT. Then

$$\Delta S = x_r(y_{r+1} - y_r) = px_r(x_{r+1}{}^m - x_r{}^m),$$

$$\Delta T = y_r(x_{r+1} - x_r) = px_r{}^m(x_{r+1} - x_r),$$

$$\frac{\Delta S}{\Delta T} = \frac{x_{r+1}{}^m - x_r{}^m}{x_r{}^{m-1}(x_{r+1} - x_r)} = \frac{x_r{}^{m-1} + x_r{}^{m-2}x_{r+1} + x_r{}^{m-3}x_{r+1}{}^2 + \cdots + x_{r+1}{}^{m-1}}{x_r{}^{m-1}}$$

$$= 1 + \left[1 + \frac{x_{r+1} - x_r}{x_r}\right] + \left[1 + \frac{x_{r+1}{}^2 - x_r{}^2}{x_r{}^2}\right] + \cdots + \left[1 + \frac{x_{r+1}{}^{m-1} - x_r{}^{m-1}}{x_r{}^{m-1}}\right]$$

$= m + hF^2$, since each of the binomials is positive and divisible by $x_{r+1} - x_r$ which is equal to h.

Now as n is increased indefinitely, h approaches 0, hence $\Delta S/\Delta T$ approaches m, the sum of all the ΔS's approaches S, the sum of all the ΔT's approaches T, hence in the limit

$$S = m \cdot T.$$

Adding first mS and then T to both sides of the equation we find

$$(1 + m)S = m(S + T), \qquad S + T = (1 + m)T,$$

hence

$$S = \frac{m}{1 + m}(S + T), \qquad T = \frac{1}{1 + m}(S + T).$$

Now obviously $S + T = xy - x_0y_0 = p(x^{m+1} - x_0^{m+1})$, so that we may write

$$S = \frac{mp}{1 + m}(x^{m+1} - x_0^{m+1}), \qquad T = \frac{p}{1 + m}(x^{m+1} - x_0^{m+1}).$$

If the foregoing proof seems to lack rigor, the objection may be easily removed as follows. We have shown that $\Delta S/\Delta T = m + hF^2$ and $\Delta T = phx_r^m$, hence

$$\Delta S = m \cdot \Delta T + hF^2 \cdot \Delta T = m \cdot \Delta T + ph^2F^2x_r^m,$$

$$\Sigma \Delta S = m \cdot \Sigma \Delta T + ph^2\Sigma(F^2x_r^m).$$

In the limit $\Sigma\Delta S = S$, $\Sigma\Delta T = T$, hence in the limit

$$S = mT + \lim [ph^2\Sigma(F^2x_r^m)],$$

and it remains to show that the second term on the right is 0.

Let G^2 represent the greatest of the n values of $F^2x_r^m$, then $\Sigma(F^2x_r^m) < n \cdot G^2$, and $ph^2\Sigma(F^2x_r^m) < ph^2nG^2 = ph(x - x_0)G^2$, since $hn = x - x_0$, and therefore

$$\lim [ph^2\Sigma(F^2x_r^m)] = \lim [ph(x - x_0)G^2] = p(x - x_0)G^2 \cdot \lim h = 0.$$

In the case of the common parabola, $m = 2$, $F^2 = 1/x_r$, and $G^2 = x$. In that case, too, the foregoing proof holds for any segment of the parabola. For if the line through the middle point of the chord and parallel to the axis of the parabola be chosen for the y-axis, and the tangent parallel to the chord for the x-axis, the equation of the parabola remains unchanged, and areas of the parallelograms corresponding to ΔS and ΔT are equal to $\Delta S \cdot \sin\theta$ and $\Delta T \cdot \sin\theta$ respectively, θ being the angle between the coördinate axes. The ratio of these areas remains therefore unchanged and the conclusion regarding the ratio of S to T remains valid.

II. Geometrical Construction of the Roots of a Cubic, and Inscription of a Regular Heptagon in a Circle.

By C. B. Haldeman, Ross, Butler County, Ohio.

1. The equation

$$q(x^2 - y^2) + 2x\sqrt{-(r^2 + q^3)} + 2ry = 0 \tag{1}$$

represents a real equilateral hyperbola when q^3 is negative and numerically greater than r^2, and

$$x^2 + y^2 = -4q \tag{2}$$